

POSITIVITY FOR GAUSSIAN GRAPHICAL MODELS

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ABSTRACT. Gaussian graphical models are parametric statistical models for jointly normal random variables whose dependence structure is determined by a graph. In previous work, we introduced trek separation, which gives a necessary and sufficient condition in terms of the graph for when a subdeterminant is zero for all covariance matrices that belong to the Gaussian graphical model. Here we extend this result to give explicit cancellation-free formulas for the expansions of nonzero subdeterminants.

1. INTRODUCTION

Gaussian graphical models are parametric statistical models for jointly normal random variables whose dependence structure is determined by a graph. In this work, we consider Gaussian graphical models on mixed graphs which have both directed edges and bidirected edges. In the causal modeling framework for graphical models [8], the directed edges represent direct causal effects of one variable on another, while the bidirected edges represent the effects of unobserved confounders that lead to correlations between the error terms of the variables. In other contexts, the Gaussian graphical models we study are known as linear structural equation models, and the directed edges describe linear relationships between variables with correlated errors.

A fundamental problem in the study of graphical models is to characterize the distributions that can arise. The models are typically presented using parametric descriptions, and we wish to give conditions on the distributions which could appear for some choice of parameters. In the case of graphical models which only have directed edges without directed cycles (so-called *directed acyclic graphs* or DAGs), it is well-known that a probability distribution belongs to the model if and only if the distribution satisfies all the conditional independence constraints implied by the graph [6, Thm 3.27].

For Gaussian graphical models, the model consists of all covariance matrices $\Sigma \in PD_m$ which arise for some choice of parameters. Each conditional independence statement $X_A \perp\!\!\!\perp X_B | X_C$, translates to a rank condition on a submatrix of the covariance matrix Σ , namely that the matrix $\Sigma_{A \cup C, B \cup C}$ has rank $\leq \#C$. (Note that for sets I and J , $\Sigma_{I,J}$ denotes the submatrix of Σ with row index set I and column index set J .) In other words, all $(\#C + 1) \times (\#C + 1)$ subdeterminants of $\Sigma_{A \cup C, B \cup C}$ are zero.

In previous work [9], we generalized the rank dropping condition from conditional independence to arbitrary minors, via *trek separation*. This gives necessary and sufficient

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conditions in terms of the underlying graph G , and sets A and B , such that $\det \Sigma_{A,B} = 0$. These vanishing determinantal constraints are among the most natural to study. In the present paper, we extend the result of [9] to give explicit cancellation-free expansions for the non-vanishing determinants $\det \Sigma_{A,B}$. This has potential applications to the algebraic study of graphical models including to the identifiability problem for Gaussian graphical models [1, 3]. Besides this, our formulas for $\det \Sigma_{A,B}$ involve elegant combinatorics in the spirit of classical enumerative combinatorics results on determinants of matrices associated to graphs [4, 7]. For example, our results give the expansion of $\det \Sigma_{A,B}$ in terms of *nonintersecting trek flows*, and we show that after collecting terms, every monomial in the expansion appears with a coefficient of the form $\pm 2^c$ for some c , determined by combinatorial properties of the flow.

This paper is organized as follows. In the next section, we give an introduction to Gaussian graphical models and the special “paths” in mixed graphs we need to study, called *treks*. In Section 3 we describe our main results on the expansions of the determinants of submatrices of Σ arising from a Gaussian graphical model. These results are based on a generalization of the Lindström-Gessel-Viennot Lemma for graphs with cycles, which is described in Section 4. Finally, Section 5 contains the proofs of the most general statements from Section 3 using the methods of Section 4.

2. GAUSSIAN GRAPHICAL MODELS

Let $G = (V, B, D)$ be a mixed graph with vertex set V , bidirected edge set B , and directed edge set D . Thus (V, D) is a directed graph without loops or multiple edges called the *directed part* of G , and (V, B) is an undirected graph without loops or multiple edges called the *bidirected part* of G . The qualification *bidirected* stems from these edges’ statistical interpretation in graphical models. A bidirected edge between vertices i and j is denoted $i \leftrightarrow j$ and a directed edge from i to j is denoted $i \rightarrow j$. Let PD_V denote the set of $\#V \times \#V$ real symmetric positive definite matrices. Let $PD(B)$ denote the set of positive definite matrices with off-diagonal nonzero entries only in positions corresponding to bidirected edges of G , i.e.

$$PD(B) = \{\Omega = (\omega_{ij}) \in PD_V : \omega_{ij} = 0 \text{ if } i \neq j \text{ and } i \leftrightarrow j \notin B\}.$$

Let $\epsilon \in \mathbb{R}^V$ be a jointly normal random vector $\epsilon \sim \mathcal{N}(0, \Omega)$ with $\Omega \in PD(B)$. For each $i \rightarrow j \in D$, let $\lambda_{ij} \in \mathbb{R}$ be a real parameter. Define a new random vector $X \in \mathbb{R}^V$ by

$$(1) \quad X_j = \sum_{i: i \rightarrow j \in D} \lambda_{ij} X_i + \epsilon_j.$$

Let Λ be the $\#V \times \#V$ matrix with λ_{ij} in the ij position if $i \rightarrow j \in D$, and zero otherwise. Equation (1) can be rearranged as $\epsilon = (I - \Lambda)^T X$, where I is a $\#V \times \#V$ identity matrix. From this, we deduce that $X \sim \mathcal{N}(0, \Sigma)$ where

$$\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$$

provided that the matrix $I - \Lambda$ is invertible.

Given $A, B \subset V$, let $\Sigma_{A,B} = (\sigma_{a,b})_{a \in A, b \in B}$ be the submatrix of Σ with row index set A and column index set B . Suppose $\#A = \#B$. In previous work [9], we gave a

combinatorial condition for when $\det \Sigma_{A,B} = 0$. In this paper, we explain how to expand $\det \Sigma_{A,B}$ explicitly in monomials without cancellation. Henceforth, rather than thinking about the λ_{ij} and ω_{ij} as parameters, we choose to think about them as indeterminates (or polynomial variables). Hence, we are interested in expanding $\det \Sigma_{A,B}$ as a polynomial or a rational function, depending on the context.

One of the key combinatorial definitions we need is the definition of a trek.

Definition 2.1. A *trek* in G from $i \in V$ to $j \in V$ is a pair (P_L, P_R) , where P_L is a directed path from some vertex s to i and P_R is a directed path from some vertex t to j , satisfying the further requirement that if $s \neq t$, then there must be a bidirected edge $s \leftrightarrow t \in B$. The vertices i and j are called the *initial vertex* and the *final vertex* of the trek, respectively.

In this definition, a path is an ordered sequence of edges in D in which the head of each edge equals the tail of the next edge. A path may have no edges, in which case, we also specify where the path starts, which is also where it ends; we call such a path an empty path. Later in the paper we will work mostly with self-avoiding paths, in which each vertex is the head of at most one edge of the path and the tail of at most one edge of the path; such a path is uniquely determined by its edge set (and initial vertex when empty). For the moment, however, P_L and P_R need not be self-avoiding. Note also that either (or both) of the paths P_L and P_R are allowed to be an empty path, in which case $s = i$ or $t = j$, respectively (or both). Two treks $T = (P_L, P_R)$ and $T' = (P'_L, P'_R)$ are the same if both $P_L = P'_L$ and $P_R = P'_R$. In particular, two treks might contain exactly the same set of edges (even with the same multiplicities) but be different treks.

If all directed edges in G are pointing down the page—which they can be arranged to do if the directed part of G happens to be acyclic—then one can think of a trek between i and j as a path of edges that starts at i , goes up for a time to s , possibly traverses a single bidirected edge to a vertex t , and then turns around and goes down for a time to j ; see Figure 1 below. We stress, however, that our results below are not restricted to the case where the directed part of G is acyclic.

To any trek $T = (P_L, P_R)$, we associate the *trek monomial* $m_T = \lambda^L \omega_{st} \lambda^R$, where $\lambda^L = \prod_{k \rightarrow l \in P_L} \lambda_{kl}$, and similarly, $\lambda^R = \prod_{k \rightarrow l \in P_R} \lambda_{kl}$. Here s, t are the initial vertices of P_L, P_R , and a variable λ_{kl} appears in λ^L and λ^R with its respective multiplicities in the paths P_L and P_R . The reason for introducing treks and trek monomials is the following result:

Proposition 2.2. *Let $G = (V, B, D)$ be a graph, and $\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$. Then for all $i, j \in V$,*

$$\sigma_{ij} = \sum_T m_T,$$

where the sum runs over all treks T in G from i to j .

Proposition 2.2 can be proven using the expansion

$$(I - \Lambda)^{-1} = I + \Lambda + \Lambda^2 + \cdots,$$

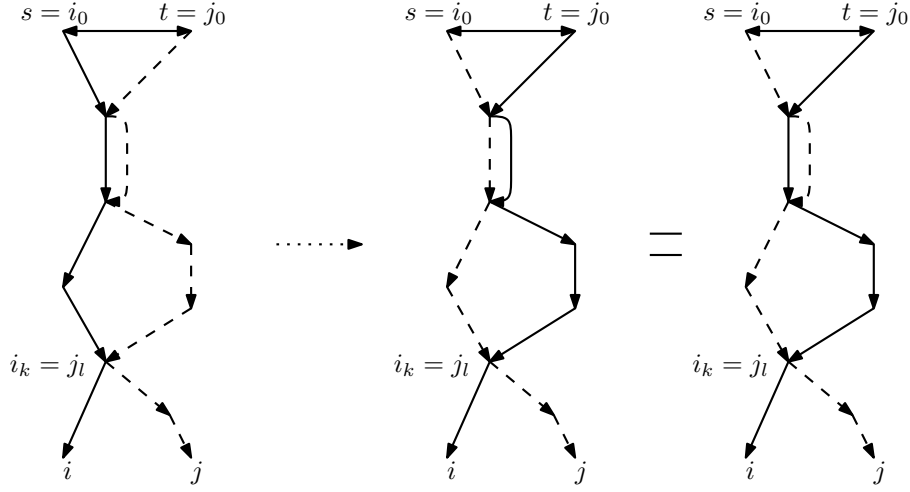


FIGURE 1. Tailswapping at (k, l) yields a trek with the same trek monomial. The two treks on the right are the same. The number of treks from i to j with this trek system monomial equals $2^{i(T)-e(T)} = 2^{3-1} = 4$.

and the combinatorial interpretation of the kl entry of Λ^r as the sum of λ^P over all directed paths P of length r from k to l in the graph (V, D) . See [9].

As stated previously, there might be many treks in $\mathcal{T}(i, j)$ that use exactly the same set of edges, so the expression for σ_{ij} might have repeated terms. As a prelude to our main results, we discuss how to simplify this expression in the case where (V, D) is acyclic. Let $T = (P_L, P_R)$ be a trek with $P_L = (s = i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_m = i)$ and $P_R = (t = j_0 \rightarrow j_1 \rightarrow \dots \rightarrow j_n = j)$. Since (V, D) is acyclic, the i_k are mutually distinct, and so are the j_l . But it may happen that i_k equals j_l , and then $T' := (P'_L, P'_R)$ with $P'_L : t = j_0 \rightarrow \dots \rightarrow j_l \rightarrow i_{k+1} \rightarrow \dots \rightarrow i_m = i$ and $P'_R : s = i_0 \rightarrow \dots \rightarrow i_k \rightarrow j_{l+1} \rightarrow \dots \rightarrow j_n = j$ is a trek with $m_{T'} = m_T$. We call this procedure *tailswapping* at (k, l) ; see Figure 1. It turns out that any trek T' with $m_T = m_{T'}$ can be obtained from T by repeated tailswapping. We call such treks *equivalent* to T , and we write $T \sim T'$. If $k = l = 0$ (so that s equals t), then clearly $T = T'$. Similarly, if $k, l > 1$ and $i_{k-1} = j_{l-1}$, then tailswapping at $(k-1, l-1)$ gives the same result as tailswapping at (k, l) . Denote by $i(T)$ the number of pairs $(k, l) \neq (0, 0)$ with $i_k = j_l$ and by $e(T)$ the number of edges that P_L and P_R have in common. The above discussion, together with the observation that tailswapping at distinct pairs of indices commutes then yields the following proposition.

Proposition 2.3. *Let $G = (V, B, D)$ be a mixed graph such that (V, D) is directed acyclic, and let $\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$. Then for all $i, j \in V$,*

$$\sigma_{ij} = \sum_{[T] \sim} 2^{i(T)-e(T)} m_T.$$

where the sum runs over equivalence classes of treks.

We will greatly generalize Proposition 2.3 in Section 3. In particular, the fact that all coefficients in $\det \Sigma_{A,B}$ are powers of 2 persists when (V, D) is a directed acyclic graph.

Now we return to the general case, where (V, D) may contain cycles. Since each entry σ_{ij} is the sum of trek monomials from i to j , the determinant $\det \Sigma_{A,B}$ will be a (signed) sum of products of trek monomials. Indeed, expanding the determinant using the standard Leibniz expansion yields the following:

Proposition 2.4. *Let $A, B \subseteq V$, with $\#A = \#B = k$. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$. Then*

$$\det \Sigma_{A,B} = \sum_{\pi \in \mathfrak{S}_k} \left(\sum_{T_1 \in \mathcal{T}(a_1, b_{\pi(1)}), \dots, T_k \in \mathcal{T}(a_k, b_{\pi(k)})} \text{sign}(\pi) \cdot m_{T_1} \cdots m_{T_k} \right).$$

There are two issues with this expansion. The first is that distinct tuples (π, T_1, \dots, T_k) can lead to the same monomial $m_{T_1} \cdots m_{T_k}$ and thus cancellation can occur. The second is that when the directed part of G contains directed cycles, this expansion is a formal power series rather than a polynomial (of course, this already happens when A and B are singletons). In the general case our aim is to write this formal power series as a rational function in which the denominator and the numerator are both in a suitable sense cancellation-free.

An important simplification that we can make when describing the graphs, matrices, and determinants is to eliminate the bidirected edges under consideration. Associated to the graph $G = (V, B, D)$, we introduce the bidirected subdivision \tilde{G} , which has only directed edges. We start with all vertices and directed edges in G . Then, for each bidirected edge $i \leftrightarrow j$ in G , we introduce a new vertex (i, j) , and two directed edges $(i, j) \rightarrow i$ and $(i, j) \rightarrow j$. The resulting graph is the bidirected subdivision of G . Combinatorial expressions for the entries σ_{ij} associated to the graph G can be obtained from the corresponding expressions for \tilde{G} as described in the following proposition.

Proposition 2.5. *Let $G = (V, B, D)$ be a graph and let \tilde{G} be its bidirected subdivision. Write $\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$ for the covariance matrix for the graph G and $\Sigma^* = (I - \Lambda^*)^{-T} \Omega^* (I - \Lambda^*)^{-1}$ for the covariance matrix for the bidirected subdivision \tilde{G} . Let $f \in \mathbb{R}[\sigma]$ be a polynomial. Then $f(\sigma(\lambda, \omega)) = 0$ if and only if $f(\sigma^*(\lambda^*, \omega^*)) = 0$.*

Furthermore, the expansion of $f(\sigma(\lambda, \omega))$ can be recovered from the expansion of $f(\sigma^(\lambda^*, \omega^*))$ by the following procedure:*

- (1) Remove any monomial that contains a $(\lambda_{(i,j),i}^*)^2$ or $(\lambda_{(i,j),j}^*)^2$.
- (2) Set all remaining $\lambda_{(i,j),i}^*$ and $\lambda_{(i,j),j}^*$ variables to 1.
- (3) Change $\omega_{(i,j),(i,j)}^*$ to ω_{ij} .
- (4) Change all remaining λ_{ij}^* and ω_{ii}^* to λ_{ij} and ω_{ii} , respectively.

Proof. Consider the \mathbb{R} -algebra homomorphism $\phi : \mathbb{R}[[\lambda, \omega]] \rightarrow \mathbb{R}[[\lambda^*, \omega^*]]$ defined by

$$\phi(\lambda_{ij}) = \lambda_{ij}^*, \quad \phi(\omega_{ii}) = \omega_{ii}^* + \sum_{j: i \leftrightarrow j \in B} (\lambda_{(i,j),i}^*)^2 \omega_{(i,j),(i,j)}^*, \quad \phi(\omega_{ij}) = \lambda_{(i,j),i}^* \omega_{(i,j),(i,j)}^* \lambda_{(i,j),j}^*.$$

We claim that $\sigma_{ij}^* = \phi(\sigma_{ij})$.

First, we prove the claim. Let Λ^* and Ω^* be the matrices for the bidirected subdivision \tilde{G} . We realize both matrices as block matrices with the first set of row/column labels

corresponding to bidirected edges $i \leftrightarrow j$, and hence the vertices (i, j) in \hat{G} . Then Λ^* and Ω^* have the form

$$\Lambda^* = \begin{pmatrix} 0 & E \\ 0 & \Lambda \end{pmatrix} \quad \Omega^* = \begin{pmatrix} \Omega_1^* & 0 \\ 0 & \Omega_2^* \end{pmatrix}$$

where $\Omega_1^* = \text{diag}((\omega_{(i,j),(i,j)})_{i \leftrightarrow j \in B})$, $\Omega_2^* = \text{diag}((\omega_{ii})_{i \in V})$, Λ is the directed edge matrix from the graph G , and E is a matrix whose $(i, j), k$ entry is $\lambda_{(i,j),k}$ if $k \in \{i, j\}$, and is zero otherwise. Note that

$$(I - \Lambda^*)^{-1} = \begin{pmatrix} I & (I - \Lambda)^{-1}E \\ 0 & (I - \Lambda)^{-1} \end{pmatrix}$$

from which we deduce that

$$(I - \Lambda^*)^{-T} \Omega^* (I - \Lambda^*)^{-1} = \begin{pmatrix} \Omega_1^* & \Omega_1^* E (I - \Lambda)^{-1} \\ (I - \Lambda)^{-T} E^T \Omega_1^* & (I - \Lambda)^{-T} (\Omega_2^* + E^T \Omega_1^* E) (I - \Lambda)^{-1} \end{pmatrix}.$$

The entry in the bottom right-hand corner of this matrix is the expression for the submatrix $\Sigma_{V,V}^*$, so the claim is equivalent to saying that the map ϕ is that map that takes Ω in $(I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$ and replaces it with $\Gamma = \Omega_2^* + E^T \Omega_1^* E$. But it is easy to see that $\gamma_{ii} = \phi(\omega_{ii})$ and for $i \neq j$, $\gamma_{ij} = 0$ if $i \leftrightarrow j \notin B$ and $\gamma_{ij} = \phi(\omega_{ij})$ if $i \leftrightarrow j \in B$.

Now we will show that the claim proves the proposition. First of all, the map ϕ is injective. Indeed, suppose that $\phi(f) = 0$ and f is irreducible. Then, since ω_{ii}^* only appears in $\phi(\omega_{ii})$, then ω_{ii} could not appear in f . Similarly, we can then rule out ω_{ij} appearing in f , and λ_{ij} appearing in f . This forces f to be the zero polynomial. Since ϕ is injective, we deduce the first part of the proposition, a power series $f(\sigma(\lambda, \omega))$ is zero if and only if $\phi(f) = f(\sigma^*(\lambda^*, \omega^*))$ is zero.

For the second part, we note that the following map ψ applied to $f(\sigma^*(\lambda^*, \omega^*))$ produces $f(\sigma(\lambda, \omega))$:

$$\psi(\lambda_{ij}^*) = \lambda_{ij}, \quad \psi(\omega_{ii}^*) = \omega_{ii} - \sum_{j: i \leftrightarrow j \in B} \omega_{ij}, \quad \psi(\omega_{(i,j),(i,j)}^*) = \omega_{ij}, \quad \psi(\lambda_{(i,j),i}^*) = 1.$$

That can be seen by applying the operations on the factorization $(I - \Lambda)^{-T} (\Omega_2^* + E^T \Omega_1^* E) (I - \Lambda)^{-1}$ to produce $(I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$. This corresponds to the four-step procedure from the statement of the proposition, since the expressions involving $(\lambda_{(i,j),i}^*)^2 \omega_{(i,j),(i,j)}^*$ will cancel with terms coming from $\psi(\omega_{ii}^*) = \omega_{ii} - \sum_{j: i \leftrightarrow j \in B} \omega_{ij}$. \square

Henceforth, we focus solely on the case of graphs with no bidirected edges.

3. RESULTS

In this section, we assume $G = (V, D)$ is a directed graph, possibly obtained as the bidirected subdivision of some mixed graph. Our goal is to find a cancellation-free analogue of the formula for $\det \Sigma_{A,B}$ in Proposition 2.4. The exact meaning of cancellation-free depends on the context. If G is acyclic, then $\det \Sigma_{A,B}$ is a polynomial in the variables ω_{ii} and the variables λ_{ij} for directed edges $i \rightarrow j$, and cancellation-free means that we determine which monomials have non-zero coefficients in $\det \Sigma_{A,B}$, and what those coefficients are.

If, on the other hand, G contains directed cycles, then there may be infinitely many treks from i to j . In this case $\det \Sigma_{A,B}$ is a formal power series representing a rational function (as is clear from $\Sigma = (I - \Lambda)^{-T} \Omega (I - \Lambda)^{-1}$ using Cramer's rule). We will then write $\det \Sigma_{A,B}$ as a fraction $\frac{R}{S}$ where R and S are polynomials with known monomials and known coefficients.

The formula that we derive in the general case has $S = 1$ when specialized to the acyclic case, hence our proof will immediately focus on the general case. For the purpose of exposition, however, we first present the formula for the acyclic case, before dealing with the general case, which is more combinatorially complicated.

3.1. Directed acyclic graphs.

Definition 3.1. Let A and B be sets of k vertices. A *trek system* \mathbf{T} from A to B consists of k treks whose initial vertices exhaust the set A and whose final vertices exhaust the set B .

Such a trek system gives rise to a bijection $A \rightarrow B$ mapping a to the final vertex of the (unique) trek in \mathbf{T} starting at a . Given a linear ordering a_1, \dots, a_k of A and a linear ordering b_1, \dots, b_k of B , this bijection determines an element π of \mathfrak{S}_k (which depends on the linear orderings of A and B), and we define $\text{sign } \mathbf{T} := \text{sign } \pi$. We define the *trek system monomial* $m_{\mathbf{T}}$ as the product of the trek monomials m_T ranging over $T \in \mathbf{T}$. The formula in Proposition 2.4 can then be rewritten as

$$(2) \quad \det \Sigma_{A,B} = \sum_{\mathbf{T}} \text{sign}(\mathbf{T}) m_{\mathbf{T}}$$

where \mathbf{T} runs over all trek systems from A to B . Cancellation can happen if distinct $m_{\mathbf{T}}$ and $m_{\mathbf{T}'}$ are the same. To achieve a cancellation-free formula we introduce the following notion (see [9]).

Definition 3.2. A trek system \mathbf{T} from A to B has *no sided intersection* if the left parts P_L for $(P_L, P_R) \in \mathbf{T}$ are mutually vertex-disjoint and also the right parts P_R for $(P_L, P_R) \in \mathbf{T}$ are mutually vertex-disjoint (but any P_L may have vertices in common with any P'_R). Otherwise, \mathbf{T} is said to have a sided intersection. We write $\mathcal{T}(A, B)$ for the set of trek systems from A to B with no sided intersection.

In [9] it is proved that in (2) one may restrict \mathbf{T} to run over all trek systems without sided intersection. Our first new result is that these terms all have the same sign and thus do not cancel.

Theorem 3.3 (Positivity for acyclic digraphs). *Let $G = (V, D)$ be an acyclic digraph and let A and B be subsets of V of the same cardinality. Assume we have fixed linear orderings of A and B . If two trek systems \mathbf{T} and \mathbf{T}' from A to B without sided intersection satisfy $m_{\mathbf{T}} = m_{\mathbf{T}'}$, then $\text{sign}(\mathbf{T}) = \text{sign}(\mathbf{T}')$.*

To obtain the desired cancellation-free formula we therefore only need to compute the number of trek systems \mathbf{T}' with the same trek system monomial as \mathbf{T} . For this we introduce combinatorial objects that we call *up-down cycles* in \mathbf{T} . Rather than giving a formal definition we describe their construction; see Figure 2 for an example. Let $x_0 \rightarrow y$

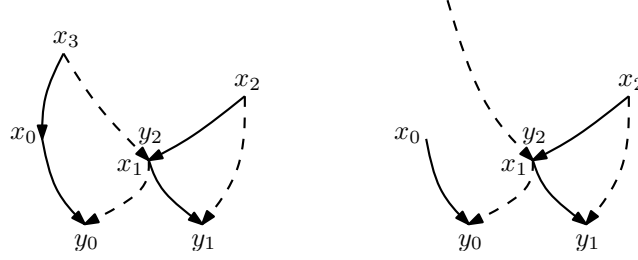


FIGURE 2. Solid directed edges denote parts of left paths in \mathbf{T} , and dashed directed edges denote parts of right paths in \mathbf{T} . On the left an up-down cycle is created, on the right, we traverse a right path up without hitting a further left path.

be an edge in some left path P_L of \mathbf{T} , and assume that $x_0 \rightarrow y$ is *not* contained in any right path of \mathbf{T} . Follow the directed path P_L *downwards*, i.e. with the direction of the graph, starting with $x_0 \rightarrow y$; since \mathbf{T} has no sided intersection, we will never intersect another left path. If no vertex on P_L after the edge $x_0 \rightarrow y$ is in any right path, then $x_0 \rightarrow y$ is not in any up-down cycle. Otherwise, let y_0 be the first vertex on P_L after $x_0 \rightarrow y$ (possibly y_0 equals y) that is also on some (necessarily unique) right path P'_R of a trek $(P'_L, P'_R) \in \mathbf{T}$. Then follow P'_R *upwards*, i.e. against the direction of the graph, from y_0 . If we never intersect another vertex on any left path (different from y_0), then again $x_0 \rightarrow y$ is not in any up-down cycle. Otherwise, let x_1 be the first vertex encountered when traversing P'_R upwards that is contained in any left path from \mathbf{T} ; say P''_L . Then follow P''_L down from x_1 , etc. Continuing in this manner we construct a sequence of non-empty paths $x_0 \rightarrow y_0 \leftarrow x_1 \rightarrow \dots$ in G that are alternatingly contained in left paths and right paths of \mathbf{T} and that do not contain edges shared by some left path with some right path (because we always turn at the *first* possible vertex in each step). Only three things can happen: we traverse a left path downward to its final vertex without hitting a further right path, we traverse a right path upward to its initial vertex without hitting a further left path, or we eventually follow the original path P_L from somewhere above the edge $x_0 \rightarrow y$. In the final case, the sequence of paths closes up and the union of their edge sets is what we call an *up-down cycle* in \mathbf{T} . The set of all up-down cycles in is denoted $\text{UD}(\mathbf{T})$. For a more formal definition see the next subsection on the general directed case.

Theorem 3.4 (Power-of-two phenomenon for acyclic digraphs). *Let $G = (V, D)$ be an acyclic digraph and let A and B be subsets of V of the same cardinality, and assume we have fixed linear orderings of A and B , as in Theorem 3.3. Let \mathbf{T} be a trek system in G without sided intersection. Then the number of trek systems $\mathbf{T}' \in \mathcal{T}(A, B)$ with $m_{\mathbf{T}'} = m_{\mathbf{T}}$ equals $2^{|\text{UD}(\mathbf{T})|}$.*

A consequence of the previous two theorems is the following formula.

Corollary 3.5 (Cancellation-free formula for acyclic digraphs). *Let $G = (V, D)$, A , and B be as in Theorem 3.3. Then*

$$\det \Sigma_{A,B} = \sum_{[\mathbf{T}] \sim \in \mathcal{T}(A,B)/\sim} \text{sign}(\mathbf{T}) 2^{|\text{UD}(\mathbf{T})|} m_{\mathbf{T}}$$

where the sum runs over equivalence classes of the relation \sim defined by $\mathbf{T} \sim \mathbf{T}'$ if and only if $m_{\mathbf{T}} = m_{\mathbf{T}'}$.

3.2. General directed graphs. For a general digraph $G = (V, D)$ that may contain cycles, the analogues of Theorems 3.3 and 3.4 and Corollary 3.5 require some notions from [10].

Definition 3.6. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be k -subsets of V with fixed linear orderings. A *self-avoiding flow* from A to B in G is a pair $\mathbf{F} = (\mathbf{P}, \mathbf{C})$ where \mathbf{P} is a set of k self-avoiding and pairwise vertex-disjoint paths whose initial vertices exhaust A and whose final vertices exhaust B , and where \mathbf{C} is a set of non-empty, self-avoiding directed cycles that are pairwise vertex-disjoint and that are also vertex-disjoint from all paths in \mathbf{P} . The path component \mathbf{P} gives rise to a bijection $A \rightarrow B$, which through the linear orderings corresponds to an element $\pi \in \mathfrak{S}_k$. The *sign* $\text{sign } \mathbf{F}$ is defined as $\text{sign}(\pi) \cdot (-1)^{|\mathbf{C}|}$, where $|\mathbf{C}|$ denotes the number of cycles in the collection \mathbf{C} .

In this definition a self-avoiding cycle is identified with its set of edges (without distinguished initial vertex). We introduce a trek analogue of this notion as follows.

Definition 3.7. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be k -subsets of V with fixed linear orderings. A *self-avoiding trek flow* in the digraph G is a pair $\mathbf{T} = (\mathbf{F}_L, \mathbf{F}_R)$ where \mathbf{F}_L is a self-avoiding flow from a (necessarily unique) k -subset S of V to A , and \mathbf{F}_R is a self-avoiding flow from the same set S to B . We call S the set of *tops* of the trek flow. We write $\mathcal{T}(A, B)$ for the set of self-avoiding trek flows from A to B .

Note that our notation \mathbf{T} and $\mathcal{T}(A, B)$ is consistent with the acyclic case: there the cycle components \mathbf{C}_L and \mathbf{C}_R of \mathbf{F}_L and \mathbf{F}_R are empty, and the condition that the path component of \mathbf{F}_L (respectively, \mathbf{F}_R) consists of self-avoiding and pairwise vertex-disjoint paths is equivalent to the condition that a trek system has no sided intersection.

Definition 3.8. The *sign* of a self-avoiding trek flow $\mathbf{T} = (\mathbf{F}_L, \mathbf{F}_R)$ from A to B (relative to fixed linear orderings on these sets) is defined as $\text{sign}(\mathbf{T}) = \text{sign } \mathbf{F}_R \cdot \text{sign } \mathbf{F}_L$. (This depends on the orderings of A and B , but not on the linear ordering of the set of tops, as long as the same ordering is used for determining the signs of both \mathbf{F}_R and \mathbf{F}_L).

The *trek flow monomial* $m_{\mathbf{T}}$ is defined as the product of the variables ω_{ii} for $i \in S$, the variables λ_{ij} corresponding to all edges $i \rightarrow j$ used by paths and cycles in \mathbf{F}_L and \mathbf{F}_R , taken with the appropriate multiplicities.

Note that, in particular, the fact that \mathbf{T} is self-avoiding implies that each λ_{ij} appears with degree at most 2 in $m_{\mathbf{T}}$. Using a theorem by the third author [10], we will write $\det \Sigma_{A,B}$ as a rational function where both the numerator and the denominator are signed sums over self-avoiding trek flows; see below. The following theorem says that no cancellation occurs.

Theorem 3.9 (Positivity for general digraphs). *Let $G = (V, D)$ be a digraph and let A and B be subsets of V of the same cardinality, both with fixed linear orderings. If two self-avoiding trek flows \mathbf{T} and \mathbf{T}' from A to B satisfy $m_{\mathbf{T}} = m_{\mathbf{T}'}$, then $\text{sign}(\mathbf{T}) = \text{sign}(\mathbf{T}')$.*

Next we compute the number of trek flows \mathbf{T}' with the same trek flow monomial as \mathbf{T} . Again, our main combinatorial tools are up-down cycles, which we define more formally at this point.

Definition 3.10. Let $\mathbf{T} = (\mathbf{F}_L, \mathbf{F}_R)$ be a self-avoiding trek flow, and let E be the set of edges used by \mathbf{F}_L or by \mathbf{F}_R but *not* by both. Define a directed graph Γ with vertex set E as follows: an element $e = x \rightarrow y \in E$ used by \mathbf{F}_L has at most one outgoing arrow in Γ . If y is visited by some path or cycle in \mathbf{F}_R *and* has an incoming edge $f = z \rightarrow y$ there, then the arrow from e points to f . If y is not visited by any path or cycle in \mathbf{F}_R (not even by an empty path based at y) *and* has an outgoing edge $f = y \rightarrow z$ in \mathbf{F}_L , then the arrow from e points to f . In all other cases, e has no outgoing arrow in Γ . Similarly, an element $e = x \rightarrow y \in E$ used by \mathbf{F}_R has at most one outgoing arrow. If x is visited by \mathbf{F}_L *and* has an outgoing edge $f = x \rightarrow z$ there, then the arrow from e points to f . If x is not visited by \mathbf{F}_L *and* has an incoming arrow $f = z \rightarrow x$ in \mathbf{F}_R , then the arrow from e points to f . In all other cases, e has no outgoing arrow.

Now an *up-down cycle* in \mathbf{T} is the support $F \subseteq E \subseteq D$ of any non-empty directed cycle in the digraph Γ . We write $\text{UD}(\mathbf{T})$ for the set of all up-down cycles in \mathbf{T} .

In fact, the definition of Γ forces every element of E to have at most one incoming arrow, as well. As a consequence, Γ is a union of vertex-disjoint directed cycles and paths. This implies that distinct up-down cycles in \mathbf{T} do not share edges, a fact that we will use later. It is easily verified that this definition of up-down cycle agrees with the construction given in the acyclic case. One new aspect is that the edges of a directed cycle in the cycle component of \mathbf{F}_L which avoid \mathbf{F}_R (or vice versa) also form an up-down cycle.

Theorem 3.11 (Power-of-two phenomenon for general digraphs). *Let $G = (V, D)$ be a digraph and let A and B be subsets of V of the same cardinality, both with fixed linear orderings, as in Theorem 3.9. Let \mathbf{T} be a self-avoiding trek flow in G . Then the number of trek flows $\mathbf{T}' \in \mathcal{T}(A, B)$ with $m_{\mathbf{T}'} = m_{\mathbf{T}}$ equals $2^{|\text{UD}(\mathbf{T})|}$.*

Finally, we state our general cancellation-free rational expression for $\det \Sigma_{A,B}$.

Corollary 3.12 (Cancellation-free formula for general digraphs). *Let $G = (V, D)$, A , and B be as in Theorem 3.9. Then*

$$\det \Sigma_{A,B} = \frac{\sum_{[\mathbf{T}] \sim \in \mathcal{T}(A,B)/\sim} \text{sign}(\mathbf{T}) 2^{|\text{UD}(\mathbf{T})|} m_{\mathbf{T}}}{\sum_{[\mathbf{T}] \sim \in \mathcal{T}(\emptyset, \emptyset)/\sim} \text{sign}(\mathbf{T}) 2^{|\text{UD}(\mathbf{T})|} m_{\mathbf{T}}},$$

where both run over equivalence classes of the relation \sim defined by $\mathbf{T} \sim \mathbf{T}'$ if and only if $m_{\mathbf{T}} = m_{\mathbf{T}'}$.

Note that every self-avoiding trek flow from the empty set to itself is of the form $((\emptyset, \mathbf{C}_L), (\emptyset, \mathbf{C}_R))$. In particular, when G is acyclic and there are no cycles, the denominator contains only one term, corresponding to $\mathbf{C}_L = \mathbf{C}_R = \emptyset$, and our formula specializes to the formula in Corollary 3.5.

4. DETERMINANTS OF PATH MATRICES FOR CYCLIC GRAPHS

In this section we recall a generalization of the Lindström-Gessel-Viennot Lemma [7] to arbitrary graphs; for a different generalization see [2]. Thus let $G = (V, D)$ be an arbitrary finite directed graph, and associate an edge variable λ_{ij} to each $i \rightarrow j \in D$. Let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be k -subsets of V , and let $M = M_{A,B}$ denote their *weighted path matrix*, i.e., the $k \times k$ -matrix over $\mathbb{R}[[\lambda_{ij} \mid i \rightarrow j \in D]]$ whose (i, j) -entry is the sum over all directed paths in G from a_i to b_j of the product of the edge variables along the path.

To express $\det(M)$ as a rational function in the edge variables, we recall from Section 3 the notion of self-avoiding flow from A to B in G , and we write $\mathcal{F}_{A,B}(G)$ for the (necessarily finite) set of all such flows.

Theorem 4.1 ([5, 10]). *The determinant of the weighted path matrix from A to B is given by*

$$\det(M_{A,B}) = \frac{\sum_{\mathbf{F} \in \mathcal{F}_{A,B}(G)} \text{sign}(\mathbf{F}) m_{\mathbf{F}}}{\sum_{\mathbf{C} \in \mathcal{F}_{\emptyset, \emptyset}(G)} \text{sign}(\mathbf{C}) m_{\mathbf{C}}},$$

where the denominator is the sum over all self-avoiding flows consisting of vertex-disjoint cycles only.

5. PROOFS OF THE MAIN RESULTS

In this section we prove Theorems 3.9 and 3.11 and Corollary 3.12, which immediately imply their acyclic special cases. Let $G = (V, D)$ be a directed graph and let $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ be k -subsets of V . We first deduce the corollary from the two theorems.

Proof of Corollary 3.12 given Theorems 3.9 and 3.11. We will apply Theorem 4.1 with G replaced by the digraph H constructed from G as follows: in the disjoint union of the opposite graph G^{opp} with G add an arrow from any vertex i^{opp} of G^{opp} to the corresponding vertex i in G . These arrows get labels ω_{ii} , while every edge $j^{\text{opp}} \rightarrow i^{\text{opp}}$ in G^{opp} gets the same label λ_{ij} as its opposite $i \rightarrow j$ in G . With this notation, the submatrix $\Sigma_{A,B}$ of our graphical model equals the path matrix $M_{A^{\text{opp}}, B}$ in Theorem 4.1. Similarly, self-avoiding trek flows in G from A to B are in bijection with self-avoiding flows in H from A^{opp} to B , and the bijection preserves signs. Thus Theorem 4.1 yields the expression

$$\det \Sigma_{A,B} = \frac{\sum_{\mathbf{T} \in \mathcal{T}(A,B)} \text{sign}(\mathbf{T}) m_{\mathbf{T}}}{\sum_{\mathbf{T} \in \mathcal{T}(\emptyset, \emptyset)} \text{sign}(\mathbf{T}) m_{\mathbf{T}}}.$$

Theorem 3.9 shows that self-avoiding trek flows from A to B with the same trek flow monomial have the same sign, and Theorem 3.11 counts the number of such trek flows in each equivalence class. This gives the desired formula

$$\det \Sigma_{A,B} = \frac{\sum_{[\mathbf{T}] \sim \mathcal{T}(A,B)/\sim} \text{sign}(\mathbf{T}) 2^{|\text{UD}(\mathbf{T})|} m_{\mathbf{T}}}{\sum_{[\mathbf{T}] \sim \mathcal{T}(\emptyset, \emptyset)/\sim} \text{sign}(\mathbf{T}) 2^{|\text{UD}(\mathbf{T})|} m_{\mathbf{T}}}.$$

□

The remainder of this section focuses on the proofs of Theorems 3.9 and 3.11. The proofs are closely intertwined and mostly contained in the main body of the text.

First we want to reduce our arguments to the case where A and B are disjoint. For this, we first modify G as follows. For each a_i, b_i , introduce copies a'_i, b'_i with edges $a_i \rightarrow a'_i, b_i \rightarrow b'_i$ and call the resulting graph G' . Let $A' = \{a'_1, \dots, a'_k\}$ and $B' = \{b'_1, \dots, b'_k\}$ inherit their linear orderings from A and B respectively. Adding the new edges gives a bijection from self-avoiding trek flows in G from A to B to self-avoiding trek flows in G' from A' to B' . For Theorem 3.9 we observe that this bijection preserves signs, and for Theorem 3.11 we observe that it also preserves the number of up-down cycles—indeed, the new edges are not part of any up-down cycle. Thus, we may drop the accents and write G, A, B, a_i, b_i for G', A', B', a'_i, b'_i , or in other words, assume A and B are disjoint subsets of V .

Next fix a self-avoiding trek flow $\mathbf{T} = (\mathbf{F}_L = (\mathbf{P}_L, \mathbf{C}_L), \mathbf{F}_R = (\mathbf{P}_R, \mathbf{C}_R))$ from A to B with set of tops $S = \{s_1, \dots, s_k\}$, and choose the linear orderings such that \mathbf{P}_L and \mathbf{P}_R connect $s_i, i \in [k]$ to a_i and to b_i , respectively, so that the path components of both flows have sign 1. We want to show that any self-avoiding trek flow \mathbf{T}' from A to B with the same trek flow monomial as \mathbf{T} has the same sign as \mathbf{T} , and that the number of such self-avoiding trek flows is $2^{|\text{UD}(\mathbf{T})|}$.

To this end, color the edges of G with 0 if they are in some path or cycle of \mathbf{F}_L and with 1 if they are used by \mathbf{F}_R . Delete edges of G without any color; these are irrelevant for what follows. Then contract all bi-colored edges in G . This may yield some additional isolated vertices (corresponding to fully contracted cycles all of whose edges are bi-colored); delete all isolated vertices. Contracting bi-colored edges and deleting resulting empty cycles and other isolated vertices gives a bijection between self-avoiding trek flows \mathbf{T}' from A to B in the original graph satisfying $m_{\mathbf{T}'} = m_{\mathbf{T}}$ and self-avoiding trek flows in the resulting graph having *full support* (i.e., containing all edges exactly once) and having S as set of tops. This bijection is sign-preserving because it does not affect the bijections $A \rightarrow B$ and because it deletes cycles in pairs. Also note that the number of up-down cycles in \mathbf{T} does not change, since up-down cycles by definition do not contain bi-colored edges. Again, we change notation so that G stands for the new graph.

Next, consider any monochromatic cycle in G that has no vertices in common with paths or cycles of the other color. This cycle can be moved freely between the left and right cycle components of a self-avoiding trek flow with full support without changing the sign or the trek flow monomial. Moving such a cycle gives a factor 2 when counting self-avoiding trek flows of full support with S as set of tops, and this factor is accounted for by the fact that the cycle counts as an up-down cycle. So it suffices to prove our theorems for the simplified graph where all such cycles (edges and vertices) are deleted from G . Update G again.

Finally, contract all vertices with exactly one incoming edge and one outgoing edge (necessarily of the same color). This can now be done without contracting a monochromatic cycle (since such cycles always intersect a cycle of the other color after we have

removed the monochromatic cycles from the preceding according to the preceding paragraph). Again, there is a sign-preserving bijection, and the number of up-down cycles is invariant.

After these modifications, the vertices in S have exactly two outgoing edges, one colored 0 and one colored 1, and no incoming edges, the vertices in $A \cup B$ have exactly one incoming edge colored 0 (for A) or 1 (for B) and no outgoing edges, and all other vertices in G have exactly two incoming edges (of different colors) and exactly two outgoing edges (again of different colors). The resulting graph, again denoted $G = (V, D)$, does not have any loops (but of course it may have directed cycles).

We can recover \mathbf{T} from the coloring by taking for \mathbf{F}_R the 0-colored edges and for \mathbf{F}_L the 1-colored edges. We may therefore identify \mathbf{T} with the $\mathbb{Z}/2\mathbb{Z}$ -coloring. More generally, self-avoiding trek flows \mathbf{T}' from A to B with the same monomial as \mathbf{T} correspond bijectively to edge colorings $D \rightarrow \mathbb{Z}/2\mathbb{Z}$ with the same properties as \mathbf{T} , namely: the incoming edges into A are colored 0, the incoming edges into B are colored 1, the outgoing edges of each vertex in $V - (A \cup B)$ are colored with distinct colors, and so are the two incoming edges of each vertex in $V \setminus (S \cup A \cup B)$. We identify such \mathbf{T}' with their colorings. Then the sum $\mathbf{T}' + \mathbf{T}$ (modulo 2) is an edge coloring where the sum of the colors of the edges entering any vertex is zero, and so is the sum of the colors leaving any vertex. Conversely, adding a coloring h with these properties to \mathbf{T} yields a self-avoiding trek flow \mathbf{T}' with the same monomial as \mathbf{T} . Since the conditions on h define a subgroup $K(\mathbf{T})$ of $(\mathbb{Z}/2\mathbb{Z})^D$ the number of self-avoiding trek flows \mathbf{T}' with the same monomial as \mathbf{T} is a power of 2. The following proposition completes the proof of Theorem 3.11.

Proposition 5.1. *To each up-down cycle C in \mathbf{T} assign an edge coloring $D \rightarrow \mathbb{Z}/2\mathbb{Z}$ that colors the edges in C with 1 and the remaining edges with 0. The resulting up-down colorings form a basis of $K(\mathbf{T})$ as a vector space over $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Let h be a non-zero element of $K(\mathbf{T})$ and let $x_0 \rightarrow y_0$ be an edge contained in the left part of \mathbf{T} that is colored 1 by h . Then y_0 has a unique other incoming edge $x_1 \rightarrow y_0$ that must also be colored 1 by h , and this edge must be in the right part of \mathbf{T} . Then x_1 has a unique other outgoing edge $x_1 \rightarrow y_2$ that must also be colored 1 by h , and this edge must be in \mathbf{T}_L , etc. All these edges are distinct, until you get back to $x_0 \rightarrow y_l$. This gives an up-down cycle whose coloring can be subtracted from h to get a coloring with fewer edges colored 1. This proves that up-down colorings span $K(\mathbf{T})$. Since distinct up-down cycles are edge-disjoint, up-down colorings are also linearly independent. \square

To prove that the signs of \mathbf{T}' and \mathbf{T} are the same, it now suffices to prove that adding to \mathbf{T} a coloring supported on an up-down cycle does not change the sign. For this we prove Lemma 5.3. To do this we first need to give a definition of signs of other combinatorial objects, which are used in the proof.

Definition 5.2. Let δ be an element of \mathfrak{S}_k , let X, Y be subsets of $[k]$ of the same cardinality, and let v be a bijection from $[k] - Y$ to $[k] - X$. We define the *sign* of the pair (δ, v) as follows. Construct a directed bipartite graph H on two copies $[k]_1, [k]_2$ of $[k]$ with δ prescribing the arrows from $[k]_1$ to $[k]_2$ (“down”) and v prescribing the arrows from $[k]_2$ (“up”). The graph H defines a bijection $\pi : X \rightarrow Y$ which maps $x \in X$ to the

endpoint of the path in H starting at X . The sign $\text{sign}(\delta, v)$ is defined as the sign of π (relative to the natural linear orderings on $X, Y \subseteq [k]$) times (-1) raised to the number of directed cycles in H .

Lemma 5.3. *Let v be as in the definition. Then $\text{sign}(\delta, v)\text{sign}(\delta)$ depends only on v and not on $\delta \in \mathfrak{S}_k$.*

Proof. We prove the lemma by induction on $k - |Y| (= k - |X|)$. It is trivially true if X and Y are both equal to $[k]$: then the graph H does not have any upward arrows and $\pi = \delta$, and $\text{sign}(\delta, v) = \text{sign}(\pi) = \text{sign}(\delta)$, so that $\text{sign}(\delta, v)/\text{sign}(\delta)$ equals 1, independently of δ .

Now suppose that the lemma is true for all $v : [k] - Y \rightarrow [k] - X$. Pick such an v , let $x \in X$ and $y \in Y$, and let v' be the extension of v to $[k] - Y + y$ sending y to x . Let $\delta \in \mathfrak{S}_k$ and construct H and π as in the definition (from the pair δ, v). To analyze how $\text{sign}(\delta, v')$ differs from $\text{sign}(\delta, v)$ we distinguish two cases:

- (1) $\pi(x) = y$: In this case, adding the upward arrow $y \rightarrow x$ to H creates an additional cycle in H , and this contributes a factor -1 to the sign. Furthermore, the new bijection $\pi' : X - x \rightarrow Y - y$ is the restriction of π to $X - x$. Hence we have

$$\text{sign}(\pi')/\text{sign}(\pi) = \text{sign} \left(\prod_{i \in X-x} (i-x)(\pi(i)-y) \right) = (-1)^{|\{i \in X | i < x\}| + |\{j \in Y | j < y\}|}.$$

In conclusion, $\text{sign}(\delta, v')/\text{sign}(\delta, v)$ is this latter expression times -1 (for the additional cycle created in H).

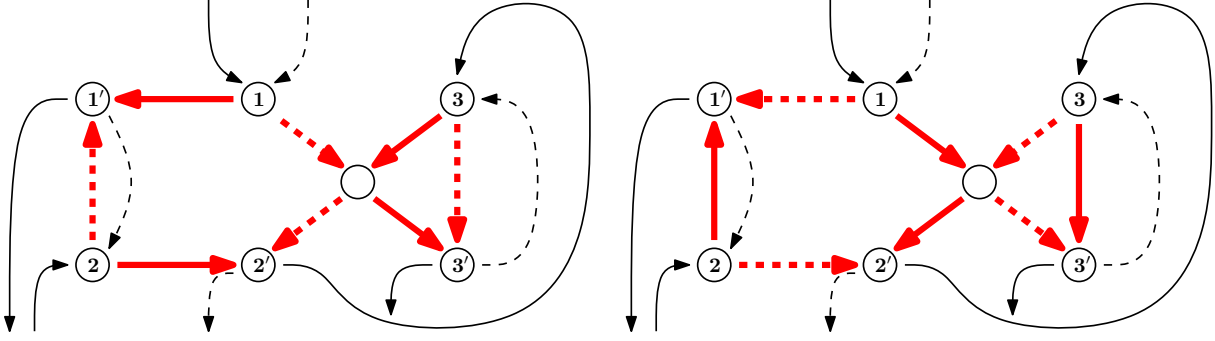
- (2) $\pi(x) = z \neq y$: In this case, no new cycle is created when adding to H the upward arrow $y \rightarrow x$. Let $u := \pi^{-1}(y)$, so that in the new graph the path starting at $u \in X$ passes through u, y, x, z in that order. Then the new permutation $\pi' : X - x \rightarrow Y - y$ equals π on $X - u - x$ and sends u to z . Hence we have

$$\begin{aligned} \frac{\text{sign}(\pi')}{\text{sign}(\pi)} &= \text{sign} \left(\frac{\prod_{i \in X-u-x} (i-u)(\pi(i)-z)}{(u-x)(y-z) \prod_{i \in X-u-x} [(i-u)(\pi(i)-y)(i-x)(\pi(i)-z)]} \right) \\ &= \text{sign} \left(\frac{1}{(u-x)(y-z) \prod_{i \in X-u-x} [(\pi(i)-y)(i-x)]} \right) \\ &= -\text{sign} \left(\frac{1}{\prod_{i \in X-x} [(i-x)] \prod_{i \in X-u} [(\pi(i)-y)]} \right) \\ &= -(-1)^{|\{i \in X | i < x\}| + |\{j \in Y | j < y\}|}. \end{aligned}$$

We conclude that the change of sign is the same in both cases, and that it does not depend on δ . \square

We will use the following direct consequence of the lemma.

Lemma 5.4. *Fix a natural number k . Let δ_0, δ_1 be elements of \mathfrak{S}_k and let v_0, v_1 be bijections between subsets $[k] - Y_0, [k] - Y_1$ and $[k] - X_0, [k] - X_1$, respectively. Hence $|X_i| = |Y_i|$ for $i = 0, 1$; we do not require that $|X_0| = |X_1|$. Then we have the following*


 FIGURE 3. Left: a fragment of \mathbf{T} ; and right: the corresponding fragment of \mathbf{T}' .

identity:

$$\text{sign}(\delta_0, v_0) \cdot \text{sign}(\delta_1, v_1) = \text{sign}(\delta_1, v_0) \cdot \text{sign}(\delta_0, v_1).$$

Now we return to our proof of positivity; the following arguments are clarified in Example 5.5 below. Suppose that \mathbf{T}' and \mathbf{T} differ by an up-down cycle h , considered as an edge coloring as above. Label the vertices with incoming edges of h -color 0 and outgoing edges of h -color 1 by the numbers $1, \dots, k$ (we identify these with a first copy $[k]_1$ of $[k]$), and similarly label the vertices with outgoing edges of h -color 0 and incoming h -color 1 by $1, \dots, k$ (identifying them with a second copy $[k]_2$). Note that there are, indeed, equally many of both, since the edges of h -color 1 and \mathbf{T} -color 0 form k vertex-disjoint paths from the first set to the second set. Denote the corresponding bijection by $\delta_0 \in \mathfrak{S}_k$. Similarly, the edges of h -color 1 and \mathbf{T} -color 1 give a bijection $\delta_1 \in \mathfrak{S}_k$. Now some directed paths with \mathbf{T} -color 0 emanating from $[k]_2$ (hence with h -color 0) may “loop back” to $[k]_1$; this gives an bijection v_0 from a subset $[k]_2 - Y_0$ of $[k]_2$ to a $[k]_1 - X_0$. Similarly, we obtain an injective map v_1 from a subset of $[k]_2 - Y_1$ into $[k]_1 - X_1$ by following directed paths of \mathbf{T} -color 1 emanating from $[k]_2$.

To change \mathbf{T} into \mathbf{T}' the colors in the up-down cycle are interchanged. This means that the roles of δ_0 and δ_1 are interchanged. The total change in trek flow sign is exactly the change from $\text{sign}(\delta_0, v_0)\text{sign}(\delta_1, v_1)$ into $\text{sign}(\delta_1, v_0)\text{sign}(\delta_0, v_1)$. Indeed, the change in sign of the left flow \mathbf{F}_L is exactly $\text{sign}(\delta_1, v_0)/\text{sign}(\delta_0, v_0)$, and the change in sign of the right flow \mathbf{F}_R is exactly $\text{sign}(\delta_0, v_1)/\text{sign}(\delta_1, v_1)$. Now Lemma 5.4 implies that that \mathbf{T} and \mathbf{T}' have the same sign. This concludes our proof of Positivity for general digraphs. We conclude the paper with an example illustrating the arguments just given.

Example 5.5. In Figure 3 we have $k = 3$, the elements of $[3]_1$ are denoted $1, 2, 3$, and the elements of $[3]_2$ are denoted $1', 2', 3'$. On the left is a fragment of \mathbf{T} with 0-colored edges solid and 1-colored edges dashed. On the right is the corresponding fragment of \mathbf{T}' .

The bijection δ_0 maps $1 \mapsto 1', 2 \mapsto 2', 3 \mapsto 3'$ and is depicted by straight solid arrows on the left and by dashed solid arrows on the right. The bijection δ_1 maps $1 \mapsto 2', 2 \mapsto 1', 3 \mapsto 3'$ and is depicted by straight dashed arrows on the left and by straight solid arrows on the right. Together these form the up-down cycle. The partial map up v_0 maps $2' \mapsto 3$ (curved solid arrow) and the partial map v_1 maps $1' \mapsto 2, 3' \mapsto 3$ (curved dashed

arrows). The remaining arrows are fragments of paths leading from the set of tops to the up-down cycle and from the up-down cycle to $A \cup B$.

The map π_0 constructed from (δ_0, v_0) as in Definition 5.2 can be read off from the left-hand picture: it maps $1 \mapsto 1', 2 \mapsto 3'$ and has sign 1; the number of cycles in the digraph H_0 as in that definition is zero. So $\text{sign}(\delta_0, v_0) = 1$. Similarly, π_1 constructed from (δ_1, v_1) is $1 \mapsto 2'$, with sign 1, and H_1 has two cycles $21'$ and $33'$, so $\text{sign}(\delta_1, v_1) = 1$, as well.

On the right, the roles of δ_0 and δ_1 are reversed. The pair (δ_1, v_0) leads to no cycles and to the map $1 \mapsto 3', 2 \mapsto 1'$ with sign -1 . Thus $\text{sign}(\delta_1, v_0) = -1$. Similarly, (δ_0, v_1) leads to the map $1 \mapsto 2'$ with sign 1 and to one cycle $33'$, so $\text{sign}(\delta_0, v_1) = -1$. Note that $\text{sign}(\delta_0, v_0)\text{sign}(\delta_1, v_1) = \text{sign}(\delta_1, v_0)\text{sign}(\delta_0, v_1)$ in accordance with Lemma 5.4, and this shows that the sign of \mathbf{T} does not change under swapping the 0 and 1-labels in the up-down cycle, resulting in \mathbf{T}' .

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